thallium. Prof. Lindemann has been good enough to examine these plates, and he is able to confirm the reality of the difference in sharpness, but it may perhaps be due to slight variations in pressure or the admixture of other elements, and it would not be safe to draw any conclusions from it. The results in the case of lead seem so definite that the spectroscope may yet prove to be a most valuable weapon in attacking the problem of isotopic elements.

The Scattering of X- and γ-Rays by Rings of Electrons—A Crucial Test of the Electron Ring Theory of Atoms.

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1. It is generally admitted that the Electron Ring Theory of Atoms, based on the classical electrodynamics and mechanics, does not explain certain results of recent investigations respecting thermal radiation, photoelectric emissions, transformations between corpuscular and X-ray energies and spectrum series. All these phenomena involve the interaction of radiation in one form or another, not merely with electrons, but also with matter, not so much with matter in bulk as with material atoms, and the failure of the Electron Ring Theory to account for them may after all be due rather to our ignorance of atomic structure than to defects in the fundamental principles of the theory itself. Moreover the hypotheses proposed for the explanation of the phenomena in question, such as the Quantum hypothesis, themselves fail to account for other phenomena, such as interference, diffraction and polarisation, which are so familiar to us that the necessity of explaining them afresh is apt to be overlooked when a new theory is introduced. In view of the present unsatisfactory state of our theories and the indecisive character of our experimental knowledge a crucial experimental test of the Electron Ring theory is very much to be desired, especially one involving as little knowledge of atomic structure as possible. The object of the present investigation is to show that the phenomenon of the scattering of X- and γ-rays by matter affords a test of this kind and, moreover, if the issue of the test be favourable, supplies a means of investigating experimentally the arrangement of the electrons in at any rate the lighter atoms. By scattering is meant the redistribution in space without change of frequency of a portion

of the radiation incident on matter. For our purpose any radiation of altered frequency, generated by processes akin to fluorescence or phosphorescence, must be carefully distinguished from scattered radiation.

2. Many years ago Sir J. J. Thomson propounded his well-known theory of the scattering of X- and γ-rays by matter, based mainly on three assumptions: (1) X- and γ-rays consist of extremely thin and intense electromagnetic pulses; (2) scattering is due entirely to the comparatively mobile electrons of the atom, that due to the relatively inert residue being neglected; and (3) the electromagnetic forces exerted by the pulses on the electrons are so intense that the forces due to other electrons and to the residue of the atom may be left out of account. Two of the results of this theory are important for the purpose of comparison with those of the present investigation: (1) The fraction of the incident energy scattered in a direction making an angle θ with the direction of propagation of the incident beam is proportional to $1+\cos^2\theta$; and (2) the energy scattered every second in all directions together is $8\pi e^4 n/3c^4 m^2$ times the intensity of the incident radiation for n electrons all of which scatter effectively, and it is independent of their arrangement. With the usual values, $e/c = 1{\cdot}59 \,.\, 10^{-20}$ and $e/cm = 1{\cdot}76 \,.\, 10^7$ *e.m.u.*, the coefficient is equal to $0{\cdot}656 \,.\, 10^{-24}$ per electron. If we assume that the hydrogen atom has a mass $1{\cdot}64 \,.\, 10^{-24}$ grm. and contains one mobile electron, we find that the mass scattering coefficient of hydrogen is 0·40. If further we assume that the number of electrons in atoms other than the hydrogen atom is one-half of the atomic weight, we find that the mass scattering coefficients of elements other than hydrogen are all equal to 0·20.

3. Recent experiments contradict this theory of scattering in three respects: (1) although the distribution of scattered radiation actually observed agrees with the $1+\cos^2\theta$ formula for directions nearly transverse to the beam, the amount of energy scattered in the forward direction exceeds, and in the backward direction falls short of that given by the formula, so that there is considerable asymmetry fore and aft, increasing with the hardness of the incident rays; (2) the atomic scattering coefficient is not equal to 0·20 for all elements other than hydrogen and for all rays, but diminishes to a very much smaller value for the most penetrating γ-rays for aluminium, iron and lead (Ishino); and (3) scattering always appears to be accompanied by a change of type corresponding to a softening of the radiation (Florance), but whether this effect is due to an admixture of softer fluorescent radiation or not, does not appear to be determined.

4. In view of this contradiction between Sir J. J. Thomson's theory and experiment I have endeavoured to develop a more complete theory of the scattering of X- and γ-rays, replacing the simple pulse theory adopted in his

first assumption by the wave theory, which has superseded it, but retaining his last two assumptions. The third assumption requires us to assume that the frequency of the incident radiation is large compared with the frequencies of the free vibrations of the mobile electrons in the atom; the limits, within which this assumption is satisfied with a sufficient approximation, can only be determined exactly by experiment in our present state of ignorance as to the structure of the atom, but a rough estimate can be got for rotating rings of electrons by using the angular velocity in place of the frequencies, since it is of the right order of magnitude according to the various theories of atomic structure proposed hitherto.

In addition to these three assumptions, we shall assume, for the purpose of the present investigation, that (4) the mobile electrons of the atom are arranged in coaxal rings of equidistant electrons, which are rotating with uniform angular velocities, usually different for different rings, though, as a particular case, one or more of the rings may be at rest; and (5) the velocity of each electron is so small compared with that of light that the effect of the magnetic forces of the waves, as well as the reaction due to radiation from the electron and the variation of its mass with speed, may be neglected. The fourth assumption is needed to ensure the permanence of the rings in spite of radiation, and is justified by the experiments of O. W. Richardson and of Einstein and de Haas on the gyrostatic momentum of magnetised substances. The fifth is made for the sake of simplicity, to avoid confusing the main lines of the theory with unnecessary detail, but there is some justification for it in the fact that theories of atomic structure, such as that of Bohr, assign values to β, the ratio of the velocity of an electron to that of light, which are only of the order 0·01 for hydrogen and other light atoms; it may possibly cease to afford a sufficient approximation for heavy atoms in which the variation of mass with speed has to be taken into account in calculations respecting the determination of their constitution from their X-ray spectra (Vegard).

5. In accordance with our first assumption, we shall treat the incident X- or γ-radiation as consisting of undamped trains of plane unpolarised waves; it may be stated at once that the theory based on this and the remaining assumptions given above is one of diffraction by the electrons, coupled together kinematically in a ring, and constituting, as it were, an electronic grating. For hydrogen it leads to the same result as the simple pulse theory; for atoms containing rings of more than one electron it gives fore and aft asymmetry of scattered radiation and a scattering coefficient, which assumes the simple pulse theory's value for waves of infinitely high frequency, but increases, as the frequency diminishes, through a series of

more and more accentuated minima and maxima to a value not much less than n times the simple pulse theory's value, where n is the number of electrons in the ring. Thus the theory does not account for the exceptionally small scattering observed by Ishino, nor for the softening of the rays found by Florance; but there is reason to suppose that both of these effects may be explained on the assumption that X- and γ-rays consist of damped wavetrains. This modification of the theory is being investigated at present.

Specification of the Incident Radiation.

6. For the sake of generality, we shall assume that the incident radiation consists of a large number of independent simultaneous, or, alternatively, successive trains of undamped, elliptically polarised, parallel, plane waves, distributed in such a way in space, or in time, as the case may be, as to produce on the whole unpolarised radiation.

Each constituent wavetrain is characterised by five parameters: (1) The frequency, ν, the same for all. (2) The epoch, ϵ, chosen so that the phase is $\nu t+\epsilon$ for the wave which at time t happens to pass through the centre of the electron ring. We shall assume that ϵ is on the average distributed uniformly over the whole range from 0 to 2π. (3) and (4) The semiaxes, a and b, of the vibration ellipse. (5) The polarisation azimuth, α, measured in the wave front from some convenient initial line to the semiaxes a. We shall suppose that α also is on the average distributed uniformly over the whole range from 0 to 2π. The assumptions respecting the distribution of ϵ and α are obviously needed to ensure the average unpolarised character of the radiation as a whole.

The electric and magnetic forces at the centre of the electron ring at time t have components, in the directions of the semiaxes OA, OB of the vibration ellipse, which are given by the equations

$$d_A = h_B = \Sigma a \cos(\nu t+\epsilon), \qquad d_B = -h_A = \Sigma b \sin(\nu t+\epsilon) \tag{1}$$

where the summation is to be taken for all possible sets of values of ϵ, a, b, and α.

The average energy flux, I, is the sum of the several component fluxes on account of the independence of the corresponding wavetrains and the average distribution of their epochs and polarisation azimuths. Hence we have by Poynting's theorem

$$\mathrm{I} = c(\overline{d_A h_B} - \overline{d_B h_A})/4\pi = \Sigma c(a^2+b^2)/8\pi \tag{2}$$

where the bar denotes time averaging, and c is the velocity of light.

Specification of the Scattering Electron Ring.

7. Let the radius of the ring be ρ, the number of equidistant electrons n, their charge e *e.s.u.*, their mass m and their uniform angular velocity ω. Also let $\beta = \omega\rho/c$; then according to assumption (5) β is so small that m may be treated as constant and used to represent the transverse or the longitudinal mass at pleasure.

The relative positions of the incident and scattered wavetrains and the electron ring are shown in fig. 1, which represents the surface of the sphere whose centre is the centre O of the electron ring, and whose radius is unity.

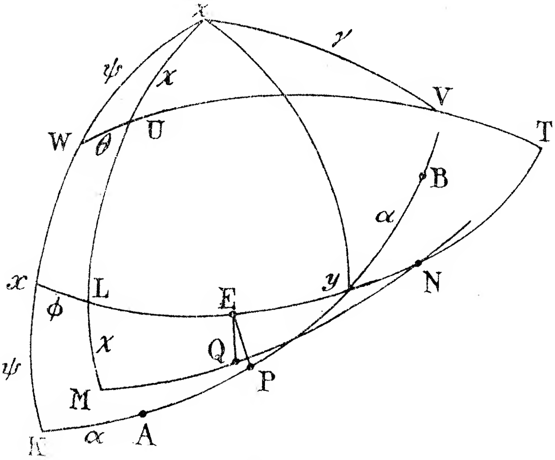

FIG. 1.

The great circle xy marks the plane of the ring, the direction of revolution being from x to y, and z marks its axis. E marks the radius vector to the ith electron at time t; we take the arc xE to be equal to $\omega t + \delta + 2\pi i/n$, where i is any integer from 0 to $n-1$.

The great circle zx has been chosen to pass through W, which marks the direction of propagation of the wavetrains of the incident radiation; Ky marks that wave front of one of the trains, which happens to pass through the centre of the ring at time t, and A, B mark the semi-axes OA, OB of the vibration ellipse, so that KA $= \alpha$.

U marks the direction in which the scattering takes place; the great circle MN marks that scattered wave front which passes through the centre of the electron ring at time t, and V marks the external bisector of the angle WOU —it is required for purposes of calculation later.

EP is an arc of the great circle WEP drawn through E perpendicular to the incident wave front KY; similarly EQ is an arc of the great circle UEQ drawn through E perpendicular to the scattered wave front MN.

8. We shall use the following notation :—

$$zW = xK = xyK = \psi, \qquad zU = LM = LNM = \chi,$$
$$zV = \gamma, \qquad WU = \theta, \qquad xL = xzL = \phi. \tag{3}$$

We shall require the perpendicular distances, p and q, of the electron E from the incident and scattered wave fronts Ky and MN respectively; these will be reckoned positive when the electron is in front of the respective wave front. Thus we obtain by means of fig. 1 and (3)

$$\left.\begin{aligned} p &= \rho\sin PE = \rho\sin Ey\sin xyK = \rho\sin\psi\cos(\omega t+\delta+2\pi i/n) \\ q &= \rho\sin QE = \rho\sin EN\sin LNM = \rho\sin\chi\cos(\omega t+\delta+2\pi i/n-\phi) \end{aligned}\right\}. \tag{4}$$

We shall require the value of $p-q$ below; we find from (4)

$$p-q = \rho\{\sin\psi-\sin\chi\cos\phi)\cos(\omega t+\delta+2\pi i/n)$$
$$-\sin\chi\sin\phi\sin(\omega t+\delta+2\pi i/n)\}. \tag{5}$$

From (3) and fig. 1 we obtain

$$\begin{aligned}(\sin\psi-\sin\chi\cos\phi)^2+\sin^2\chi\sin^2\phi &= \sin^2\psi+\sin^2\chi-2\sin\psi\sin\chi\cos\phi \\ &= 2(1-\cos\theta)-(\cos\chi-\cos\psi)^2 \\ &= 4\sin^2\gamma\sin^2\tfrac{1}{2}\theta,\end{aligned}$$

since we have

$$\cos\chi-\cos\psi = \cos\gamma(\cos UV-\cos WV) = 2\cos\gamma\sin\tfrac{1}{2}\theta.$$

Hence we find from (5)

$$p-q = 2\rho\sin\gamma\sin\tfrac{1}{2}\theta\cos(\omega t+\delta+v+2\pi i/n),$$
$$v = \tan^{-1}\frac{\sin\chi\sin\phi}{\sin\psi-\sin\chi\cos\phi}. \tag{6}$$

This result will be required presently.

The Electric Forces of the Incident and Scattered Waves.

9. In order to find the scattered radiation we must determine the electromagnetic field at a distant point due to the disturbance of the electron generated by the incident waves.

Let the field point be in the direction of the unit vector **U** at a great distance R from the centre of the electron ring; its distance from the electron E is equal to $R-q$ very approximately.

Also let **f** denote the acceleration of the electron; then we know that to the same approximation the electric and magnetic forces at the field point at time t are given by

$$\mathbf{d} = -\frac{e}{c^2R}\{\mathbf{f}-(\mathbf{Uf})\,\mathbf{U}\}_{t-(R-q)/c}, \qquad \mathbf{h} = [\mathbf{Ud}]. \tag{7}$$

Terms of the relative order q/R have been neglected in calculating *intensities,* but they must be retained in estimating *phases,* so that the function inside the bracket in the expression for $\mathbf{d}$ must be taken for the time $t-(\mathrm{R}-q)/c$, not for the time $t-\mathrm{R}/c$, in order to calculate the diffraction effect.

10. In order to calculate the acceleration $\mathbf{f}$ of the electron due to the incident waves we must bear in mind our assumptions (3), § 2, and (5), § 4. In virtue of (3) we need only retain the effective force due to the incident waves on the left of the vector equation of motion, neglecting the effective force due to steady revolution as well as the controlling forces due to the remaining electrons of the ring and the residue of the atom. In virtue of (5) we may on the left take the mass of the electron to be the constant m, and on the right neglect the effect of the magnetic forces of the incident waves. Denoting the electric force of the incident wave at the *undisturbed* position of the electron at the time of emission by $\mathbf{d}'$ for the moment, we may write

$$m\mathbf{f} = e\mathbf{d}'. \qquad (8)$$

In writing down this equation we have neglected the effect of the disturbance on the mass of the disturbed electron and have used the *undisturbed* instead of the *disturbed* position of the electron at the time of emission in calculating the electric force which generates the disturbance. The error is of the order of the square of the disturbing force in each case; we shall neglect errors of this order throughout for the sake of simplicity.

11. We must now find the value of the disturbing electric force $\mathbf{d}'$ at the undisturbed position of the electron at the time of emission. We commence by finding its components at the centre of the electron ring at the time t by means of (1), § 6, and fig. 1, § 7. Resolving the electric force in the directions OK, Ox, Oy, and Oz we obtain

$$\left.\begin{aligned} d_{\mathrm{K}}' &= \Sigma\{a\cos\alpha\cos(\nu t+\epsilon)-b\sin\alpha\sin(\nu t+\epsilon)\} \\ &\qquad\qquad = \tfrac{1}{2}\Sigma(a\cos\alpha+\iota b\sin\alpha)\exp\iota(\nu t+\epsilon) \\ d_y' &= \Sigma\{a\sin\alpha\cos(\nu t+\epsilon)+b\cos\alpha\sin(\nu t+\epsilon)\} \\ &\qquad\qquad = \tfrac{1}{2}\Sigma(a\sin\alpha-\iota b\cos\alpha)\exp\iota(\nu t+\epsilon) \\ d_x' &= d_{\mathrm{K}}\cos\psi, d_x' = -d_{\mathrm{K}}\sin\psi \end{aligned}\right\}, \qquad (9)$$

where the conjugate terms have been omitted for the sake of brevity.

Clearly (9) give the components of the electric force $\mathbf{d}'$ at the undisturbed position of the electron E at the time t, provided that we replace t in (9) by $t-p/c$, because every incident wave front reaches E at a time later by the amount p/c, where p is given by (4), § 8, for the time t.

Also (8), § 10, gives the corresponding acceleration at time t. Hence we find that the acceleration due to the disturbance of the ith electron **E** at time t is given by

$$\left.\begin{aligned} f_x &= \frac{e\cos\psi}{2m}\Sigma(a\cos\alpha+\iota b\sin\alpha)\exp\iota\{\nu(t-p/c)+\epsilon\} \\ f_y &= \frac{e}{2m}\Sigma(a\sin\alpha-\iota b\cos\alpha)\exp\iota\{\nu(t-p/c)+\epsilon\} \\ f_z &= -\frac{e\sin\psi}{2m}\Sigma(a\cos\alpha+\iota b\sin\alpha)\exp\iota\{\nu(t-p/c)+\epsilon\} \\ \text{with}\quad p &= \rho\sin\psi\cos(\omega t+\delta+2\pi i/n) \end{aligned}\right\}. \quad (10)$$

12. In calculating the electric and magnetic forces of the scattered wave by means of (7), § 9, we must not take the components of **f** for the time t when the wave arrives at the field point, but for the time $t-(R-q)/c$ when it is emitted by the scattering electron, so that we must replace t in (10), § 11, by $t-(R-q)/c$, and calculate q by (4), § 8, with $t-R/c$ in place of t. Thus in place of (10) we must use the following equations:—

$$\left.\begin{aligned} f_x &= \frac{e\cos\psi}{2m}\Sigma(a\cos\alpha+\iota b\sin\alpha)\exp\iota\{\nu(t-R/c)+\epsilon-\nu(p-q)/c\} \\ f_y &= \frac{e}{2m}\Sigma(a\sin\alpha-\iota b\cos\alpha)\exp\iota\{\nu(t-R/c)+\epsilon-\nu(p-q)/c\} \\ f_z &= -\frac{e\sin\psi}{2m}\Sigma(a\cos\alpha+\iota b\sin\alpha)\exp\iota\{\nu(t-R/c)+\epsilon-\nu(p-q)/c\} \\ p &= \rho\sin\psi\cos\{\omega(t-R/c)+\delta+2\pi i/n+\omega q/c\} \\ q &= \rho\sin\chi\cos\{\omega(t-R/c)+\delta+2\pi i/n-\phi\} \end{aligned}\right\}. \quad (11)$$

13. At this stage a difficulty requires notice, which arises from the occurrence of the term $\omega q/c$ in the argument of the cosine in p. With the usual notation of Bessel Functions, bearing in mind that $\omega\rho/c$ is equal to β, we may write

$$\begin{aligned} p &= \text{Real part of } \rho\sin\psi\exp\iota\{\omega(t-R/c)+\delta+2\pi i/n+\omega q/c\} \\ &= \text{Real part of } \rho\sin\psi\exp\iota\{\omega(t-R/c)+\delta+2\pi i/n\} \\ &\qquad \sum_{-\infty}^{\infty} J_j(\beta\sin\chi)\exp\iota j\{\omega(t-R/c)+\delta+\tfrac{1}{2}\pi+2\pi i/n-\phi\}. \end{aligned}$$

Owing to the smallness of β we may neglect all the Bessel Functions except $J_0(\beta\sin\chi)$ and replace this by unity; this amounts to neglecting the term $\omega q/c$ in the argument of p because it is of the order β.

Obviously we cannot do this with the term $\nu(p-q)/c$ in the argument of (f_x, f_y, f_z) of (11), because ν is assumed to be large in comparison with ω, but we may use (6), § 8, in place of (4), with $t-\mathrm{R}/c$ in place of t, so that we may write

$$\left.\begin{aligned} \nu(p-q)/c &= 2\tau\sin\gamma\sin\tfrac{1}{2}\theta\cos\{\omega(t-\mathrm{R}/c)+\delta+v+2\pi i/n\}\\ \text{where}\qquad \tau &= \nu\rho/c = 2\pi\rho/\lambda \end{aligned}\right\}. \qquad (12)$$

This equation, together with the first three equations (11), § 8, determines the values of the components of **f** to be substituted in (7), § 9.

14. By means of the well-known Bessel Function expansion we obtain from (12), § 13,

$$\begin{aligned} \exp\{-\iota\nu(p-q)/c\} &= \exp 2\iota\tau\sin\gamma\sin\tfrac{1}{2}\theta \\ &\qquad\qquad \sin\{\omega(t-\mathrm{R}/c)+\delta+v-\tfrac{1}{2}\pi+2\pi i/n\} \\ &= \sum_{-\infty}^{\infty}\mathrm{J}_j(2\tau\sin\gamma\sin\tfrac{1}{2}\theta) \\ &\qquad\qquad \exp\iota j\{\omega(t-\mathrm{R}/c)+\delta+v-\tfrac{1}{2}\pi+2\pi i/n\}. \qquad (13) \end{aligned}$$

Substituting from (13) in (11), § 12, we find

$$\begin{aligned} \mathbf{f} = \frac{e}{2m}\Sigma\sum_{-\infty}^{\infty}\{&\cos\psi(a\cos\alpha+\iota b\sin\alpha),\ a\sin\alpha-\iota b\cos\alpha, \\ &-\sin\psi(a\cos\alpha+\iota b\sin\alpha)\}\,\mathrm{J}_j(2\tau\sin\gamma\sin\tfrac{1}{2}\theta) \\ &\qquad \exp\iota\{(\nu+j\omega)(t-\mathrm{R}/c)+\epsilon+j(\delta+v-\tfrac{1}{2}\pi+2\pi i/n)\}. \qquad (14) \end{aligned}$$

Since the components of the unit vector **U** are $\sin\chi\cos\phi$, $\sin\chi\sin\phi$, and $\cos\chi$ respectively, whilst in fig. 1, § 7, $\cos\mathrm{KU} = \cos\psi\sin\chi\cos\phi-\sin\psi\cos\chi$, and $\cos y\mathrm{U} = \sin\chi\sin\phi$, we obtain from (14)

$$\begin{aligned} (\mathbf{Uf}) = \frac{e}{2m}\Sigma\sum_{-\infty}^{\infty}\{&\cos\mathrm{KU}(a\cos\alpha+\iota b\sin\alpha)+\cos y\mathrm{U}(a\sin\alpha-\iota b\cos\alpha)\} \\ &\mathrm{J}_j(2\tau\sin\gamma\sin\tfrac{1}{2}\theta)\exp\iota\{(\nu+j\omega)(t-\mathrm{R}/c)+\epsilon+j(\delta+v-\tfrac{1}{2}\pi+2\pi i/n)\}. \end{aligned} \qquad (15)$$

These expressions, (14) and (15), when they are substituted in (7), § 9, give the electric and magnetic forces at the distant point due to the waves scattered by the ith electron. To obtain those due to the whole ring we must sum the resulting expressions for all values of i from 0 to $n-1$, but since the forces are linear functions of the quantities **f** and (**Uf**), we obtain the same result by first performing the summation on the expressions (14) and (15), and then substituting the sums in (7). We notice that the specimen terms given in (14) and (15) involve the factor $\exp\iota 2\pi ij/n$; the sum with respect to i from 0 to $n-1$ of each such factor vanishes identically for all values of j, with the exception of those of the form kn, where k is an integer, and in these cases the exponential reduces to unity for every value of i, and the sum to n. Thus the required sums are obtained from (14) and (15) by changing j into kn and multiplying each expression by n. It is hardly worth while to write down their actual expressions, as the indications just given suffice for our purpose.

The Scattered Radiation.

15. To find the amount of the radiation scattered by the ring in the direction of the unit vector $\mathbf{U}$, we apply Poynting's theorem. From (7), § 9, we obtain for the Poynting flux due to the whole ring

$$\mathbf{P} = \frac{c}{4\pi}\Big[\mathbf{dh}\Big] = \frac{c}{4\pi}\, d^2\mathbf{U} = \frac{e^2}{4\pi c^3 \mathrm{R}^2}\left[\{\Sigma \mathbf{f}\}^2 - \{\Sigma\,(\mathbf{Uf})\}^2\right]\mathbf{U}, \tag{16}$$

where the summation indicated by the symbol Σ is with respect to i and is performed in the manner indicated at the end of § 14.

The radiation scattered per unit solid angle in the direction $\mathbf{U}$, to be denoted by the symbol P in future, is obtained by multiplying the coefficient of $\mathbf{U}$ in the expression (16) for $\mathbf{P}$ by R^2 and averaging the result with respect to the time t and the polarisation azimuth α. In the time averaging all squares and products of exponentials vanish identically with the sole exception of products of conjugate exponentials, so that the epochs ϵ and δ, as well as the angle v disappear in the process. In averaging with respect to α the product $\sin\alpha\cos\alpha$ vanishes, and the squares $\cos^2\alpha$ and $\sin^2\alpha$ reduce each to one-half on account of the unpolarised character of the incident radiation as a whole. Bearing in mind what was said at the end of § 14 as to the sums with respect to i of $\mathbf{f}$ and $(\mathbf{Uf})$, we obtain by means of (14), (15), and (16), in the first instance,

$$\mathrm{P} = \frac{e^4 n^2}{8\pi c^3 m^2}\,\Sigma \sum_{-\infty}^{\infty} (a^2+b^2)\{1-\tfrac{1}{2}(\cos^2 \mathrm{KU} + \cos^2 y\mathrm{U})\}\,\mathrm{J}^2_{kn}(2\tau \sin\gamma \sin\tfrac{1}{2}\theta). \tag{17}$$

The first summation is for all values of a and b, since the remaining parameters of the incident radiation, v, ϵ, and α, have all disappeared; thus $\Sigma\,(a^2+b^2)$ enters into (17) as a factor, and by (2), § 6, its value is $8\pi \mathrm{I}/c$. Moreover, we see from fig. 1, § 7, that

$$\cos^2 \mathrm{KU} + \cos^2 y\mathrm{U} = \sin^2 \mathrm{WU} = \sin^2\theta.$$

Hence (17) gives

$$\mathrm{P} = \mathrm{I}\,\frac{e^4 n^2}{2 c^4 m^2}(1+\cos^2\theta) \sum_{-\infty}^{\infty} \mathrm{J}^2_{kn}(2\tau\sin\gamma\sin\tfrac{1}{2}\theta). \tag{18}$$

16. The infinite summation with respect to k can be replaced by a finite one with respect to i, for we have the well-known equation

$$\mathrm{J}^2_{kn}(2\tau\sin\gamma\sin\tfrac{1}{2}\theta) = \frac{1}{\pi}\int_0^{\pi} \mathrm{J}_0(4\tau\sin\gamma\sin\tfrac{1}{2}\theta\sin\phi)\cos 2kn\phi\, d\phi.$$

Hence we obtain in succession by means of Dirichlet's discontinuous integral

$$\begin{aligned}\sum_{-\infty}^{\infty} J^2_{kn}(2\tau \sin\gamma \sin\tfrac{1}{2}\theta) &= \frac{1}{\pi}\int_0^{\pi} J_0(4\tau \sin\gamma \sin\tfrac{1}{2}\theta \sin\phi) \sum_{-\infty}^{\infty} \cos 2kn\,\phi\, d\phi \\ &= \frac{1}{\pi}\operatorname*{Lim}_{k\to\infty}\int_0^{\pi} J_0(4\tau\sin\gamma\sin\tfrac{1}{2}\theta\sin\phi)\frac{\sin(2k+1)n\phi}{\sin n\phi}\,d\phi \\ &= \frac{1}{n}\sum_0^{n-1} J_0\left(4\tau\sin\frac{\pi i}{n}\sin\gamma\sin\tfrac{1}{2}\theta\right).\end{aligned}$$

Substituting this result in (18), separating the term $i = 0$ from the rest and replacing the ratio P/I by s, we obtain for the scattering coefficient in the direction **U** the expression

$$s = \frac{e^4 n}{2c^4 m^2}(1+\cos^2\theta)\left\{1+\sum_1^{n-1} J_0\left(4\tau\sin\frac{\pi i}{n}\sin\gamma\sin\tfrac{1}{2}\theta\right)\right\}. \qquad (19)$$

Thus s, the scattering coefficient per unit solid angle is determined by (19) as a function of two variables only, viz., θ, the angle between the incident and scattered beams, and γ, the angle between the axis of the scattering electron ring and the external bisector OV of the angle θ, which is shown in fig. 1, § 7. The radius ρ of the ring and the wave-length λ of the incident vibration only enter into the expression for s implicitly through the parameter τ, which is equal to $2\pi\rho/\lambda$ by (12), § 13. The form of the result depends essentially on n, the number of electrons in the ring. When there is but one electron, $n = 1$, the sum disappears, and (19) gives

$$s = \frac{e^4}{2c^4 m^2}(1+\cos^2\theta). \qquad (20)$$

In this case, and this case alone, the scattering coefficient is independent of γ, which determines the position of the ring relative to the incident beam, and is simply proportional to $1+\cos^2\theta$, in agreement with the simple pulse theory. We have every reason for assuming that this is the case of the hydrogen atom.

Thus scattering experiments on hydrogen, either gas under pressure or liquid, with a sufficiently extended range of wave-lengths would enable us to test (20), but such experiments alone could not decide between the simple pulse theory and the present one.

17. When the argument of the Bessel Function in (19), § 16, is zero, that function reduces to unity and we obtain

$$s = \frac{e^4 n^2}{2c^4 m^2}(1+\cos^2\theta). \qquad (21)$$

This occurs in three cases:—

(i) When τ vanishes, *i.e.*, when the wave-length, λ, of the incident radiation is infinitely great in comparison with ρ, the radius of the electron ring. This case is never actually realised, because resonance sets in when λ becomes large enough, and assumption (3) is violated; but, owing to the flatness of the first maximum of the Bessel Function J_0, it can be approached very nearly, and we may therefore expect an increase of the scattering coefficient as the wave-length increases to values considerably greater than the value given by the simple pulse theory, and in fact not much less than n times that value.

(ii) When γ is 0 or π, *i.e.*, when the plane of the electron ring bisects the angle between the incident and scattered beams. This case might conceivably be realised with an assemblage of magnetic atoms by applying a strong magnetic field in the direction of the external bisector of the incident and scattered beams; at any rate, an increased scattering might be obtained, if not the full amount of n times the simple pulse theory's value.

(iii) When θ is 0, *i.e.*, in the forward direction of the incident beam, a case which merits special consideration.

Forward and Backward Scattering.

18. Putting $\theta = 0$ in (21), § 17, we obtain for the forward scattering

$$s_f = \frac{e^4 n^2}{c^4 m^2}, \tag{22}$$

which is exactly n times the simple pulse theory's value for every wave-length and all orientations of the electron ring.

Again, putting $\theta = \pi$ in (19), § 16, we obtain for the backward scattering

$$s_f = \frac{e^4 n}{c^4 m^2}\left\{1 + \sum_1^{n-1} J_0\left(4\tau \sin\frac{\pi i}{n}\sin\gamma\right)\right\}. \tag{23}$$

The effect of the sum in (23) is most pronounced when $\gamma = \pi/2$. This case might be realised approximately by using a magnetic substance in a strong magnetic field perpendicular to the incident beam, and observing the scattering in a direction as nearly as possible opposite to the direction of propagation of the incident beam.

The simplest example is given by the value $n = 2$, which probably corresponds to the helium atom; we obtain from (22) and (23)

$$s_f = \frac{4e^4}{c^4 m^2}, \qquad s_f = \frac{2e^4}{c^4 m^2}\{1 + J_0(4\tau \sin\gamma)\}. \tag{24}$$

The first and lowest minimum of $J_0(4\tau \sin\gamma)$ is equal to $-0{\cdot}403$ and is given by $4\tau \sin\gamma = 3{\cdot}83$, or $\lambda = 6{\cdot}56\,\rho \sin\gamma$. For this value of λ s_b is less than 30 per cent. of s_f, so that the asymmetry fore and aft is considerable. For greater values of n it may become still more marked.

Qualitative experiments with magnetic substances in strong magnetic fields of the kind indicated in this and the preceding section would enable us to decide between the simple pulse theory and the present one, but not between the latter and a theory based on the Parson ring electron, because the Parson ring electron gives rise to analogous magnetic effects.

Scattering due to an Assemblage of similar Electron Rings.

19. In order to calculate the average scattering of an assemblage of electron rings, we must make a further assumption (6): each electron ring scatters the incident radiation independently of all the other rings. Actually the radiation scattered by any one ring falls on others and is scattered by them; our assumption amounts to neglecting the proportion of radiation which undergoes double, triple, and in general multiple scattering. It is obvious that the effect of multiple scattering will be to diffuse the radiation and partially destroy the asymmetry produced by a single scattering, and will be greater the denser and thicker the assemblage is. Its existence might be detected experimentally by working with gases at different pressures; its effect might be made comparatively harmless by working with thin layers of material, or by using layers of varying thickness and extrapolating for infinitesimal thickness in the usual manner.

We shall only consider the simplest case, where the axes of the electron rings are on the average distributed equally in all directions. To find the average scattering $\bar{s}$ in the direction **U**, we multiply (19), § 16, by the chance that the angle γ should lie between the limits γ and $\gamma+d\gamma$, viz. $\frac{1}{2}\sin\gamma d\gamma$, and integrate with respect to γ from 0 to π. Thus we obtain, most simply by expanding the Bessel Function and integrating term by term,

$$\bar{s} = \frac{e^4 n}{2c^4 m^2}(1+\cos^2\theta)\left\{1 + \sum_{1}^{n-1} \sin\left(4\tau\sin\frac{\pi i}{n}\sin\tfrac{1}{2}\theta\right)\Big/ 4\tau\sin\frac{\pi i}{n}\sin\tfrac{1}{2}\theta\right\}. \quad (25)$$

This expression leads to similar results as regards asymmetry as (19), § 16, though, as is to be expected, they are not quite so marked. In the forward direction, $\theta = 0$, the scattering is n times the simple pulse theory's value as before, but in the backward direction, $\theta = \pi$, the Bessel Function term in (23), § 18, is replaced by $\sin\left(4\tau\sin\frac{\pi i}{n}\right)\Big/4\tau\sin\frac{\pi i}{n}$. This term, too, is oscillatory; its first and lowest minimum is equal to $-0{\cdot}217$ and corresponds to $4\tau\sin\frac{\pi i}{n} = 4{\cdot}49$. For the two-electron ring of helium the backward scattering, $\bar{s}_f$, is 39 per cent. of the forward scattering, $\bar{s}_f$, and occurs for $\lambda = 5{\cdot}60\rho$.

Scattering of an Irregular Assemblage of Rings through a Finite Solid Angle.

20. We shall calculate the total scattering through the front half of a cone of semivertical angle θ, with its axis parallel to the direction of the incident beam, and for this purpose multiply (25), § 19, by $2\pi \sin\theta d\theta$ and integrate from 0 to θ. The integration is best performed by changing the variable from θ to $u \equiv \sin\frac{1}{2}\theta$. Denoting the total scattering through the cone by $S(\theta)$, we obtain

$$\left.\begin{aligned} S(\theta) &= \frac{8\pi e^4 n}{3c^4m^2}\left\{u^6+\tfrac{3}{2}u^2(1-u^2)+\sum_1^{n-1}\frac{3}{4r^2}\left[1+\frac{1}{r^2}+\frac{3}{r^4}\right.\right. \\ &\quad \left.\left.-\left(1-2u^2+2u^4+\frac{1-6u^2}{r^2}+\frac{3}{r^4}\right)\cos 2ru-\frac{2u}{r}\left(1-2u^2+\frac{3}{r^2}\right)\sin 2ru\right]\right\} \\ \text{where} \quad u &= \sin\tfrac{1}{2}\theta, \qquad r = 2\tau\sin\frac{\pi i}{n} = \frac{4\pi\rho}{\lambda}\sin\frac{\pi i}{n} \end{aligned}\right\}. \qquad (26)$$

Adopting the notation of 'B. A. Report,' 1914, p. 87, we write

$$S_0(x) = \sin x, \quad S_1(x) = \frac{\sin x}{x}-\cos x, \quad S_2(x) = \left(\frac{3}{x^2}-1\right)\sin x-\frac{3}{x}\cos x. \qquad (27)$$

Then we may write (26) in the form

$$\begin{aligned} S(\theta) = \frac{8\pi e^4 n}{3c^4m^2}u^2\Big\{&u^4\left[1+\sum_1^{n-1}\{S_0^2(ru)+\tfrac{1}{2}S_2^2(ru)\}r^{-2}u^{-2}\right] \\ &+\tfrac{3}{2}(1-u^2)\left[1+\sum_1^{n-1}\{S_0^2(ru)+u^2S_1^2(ru)\}r^{-2}u^{-2}\right]\Big\}. \qquad (28) \end{aligned}$$

We shall consider two important particular cases of this formula.

Total Scattering.

21. The total scattering from the ring is obtained by putting $\theta = \pi$, or $u = 1$, in (28), § 20. In this case we shall omit the argument π of $S(\pi)$ as unnecessary and obtain

$$\left.\begin{aligned} S &= \frac{8\pi e^4 n}{3c^4m^2}\left\{1+\sum_1^{n-1} f\left(\frac{4\pi\rho}{\lambda}\sin\frac{\pi i}{n}\right)\right\} \\ \text{where} \quad f(x) &= \{S_0^2(x)+\tfrac{1}{2}S_2^2(x)\}x^{-2} \end{aligned}\right\}. \qquad (29)$$

For hydrogen we assume that $n = 1$, so that the sum disappears. Denoting the corresponding value by S_H, we find

$$S_H = \frac{8\pi e^4}{3c^4m^2} = 0{\cdot}656 \,.\, 10^{-24}, \qquad (30)$$

where the numerical values are the same as those used in § 2. The result is precisely that of the simple pulse theory, so that on both theories the scattering of hydrogen is independent of the wave-length, and its distribution is symmetrical fore and aft.

It is convenient to take hydrogen as the standard to which all other substances are referred.

Atomic and Mass Scattering Coefficients.

22. An atom is assumed to contain coaxal rings of equidistant mobile electrons (§ 3, assumption (3)). We shall neglect the possibility that two rings may have a constant phase relation between them, and so interfere with each other; this occurs when they have the same angular velocity, a contingency which is extremely unlikely on any of the current theories of atomic structure. Then the scattering of an atom is the sum of the scatterings of its several electron rings.

Let the atomic number be N so that

$$\mathrm{N} = \Sigma n, \tag{31}$$

where the summation is for all the rings. We obtain from (29) and (30), § 21,

$$\mathrm{S} = \mathrm{S}_{\mathrm{H}}\left\{\mathrm{N} + \Sigma n \sum_{1}^{n-1} f\left(\frac{4\pi\rho}{\lambda}\sin\frac{\pi i}{n}\right)\right\} \tag{32}$$

Experimenters often use the mass scattering coefficient, *i.e.*, the scattering per unit mass, in place of S, which denotes the scattering per atom, or atomic scattering coefficient. We shall denote the mass scattering coefficient by σ, and that of hydrogen by σ_{H}; to obtain them we must divide by the mass of the atom. Taking the mass of the hydrogen atom to be $1{\cdot}64 \,.\, 10^{-24}$ grm., we find from (30), § 21,

$$\sigma_{\mathrm{H}} = 0{\cdot}40. \tag{33}$$

Since the mass of an atom of atomic weight A is A times the mass of the hydrogen atom, we obtain from (32) and (33)

$$\sigma = \frac{0{\cdot}40}{\mathrm{A}}\left\{\mathrm{N} + \Sigma n \sum_{1}^{n-1} f\left(\frac{4\pi\rho}{\lambda}\sin\frac{\pi i}{n}\right)\right\}. \tag{34}$$

For molecules of compound substances we must add together the atomic scattering coefficients S of its constituents, and divide $\Sigma\mathrm{S}$ by $\Sigma\mathrm{A}$, in order to obtain the mass scattering coefficient.

One important result of (34) should be noticed; when the wave-length λ of the incident radiation becomes very great, the argument of the function f becomes very small, and the function itself becomes very nearly equal to unity, as we shall see below.

Hence we find

$$\operatorname*{Limit}_{n\to\infty} \sigma = \frac{0{\cdot}40}{A} \Sigma n^2. \tag{35}$$

The simple pulse theory gives $\sigma = 0{\cdot}40NA^{-1}$ for all wave-lengths, so that the present theory gives a very much larger value for long waves, although it gives the same value for very short waves.

The long wave limit cannot be reached owing to the intervention of resonance, but it can be approached pretty closely, say within 10 or 20 per cent., long before resonance becomes appreciable, so that measurements of scattering for long waves should enable us to decide between the two theories.

Before considering particular examples of the theory, we shall first consider the second particular case of formula (28), § 20.

Emergent and Returned Scattered Radiation.

23. Florance defines emergent scattered radiation as that which is scattered forwards on the whole, *i.e.*, between the limits $\theta = 0$ and $\theta = \frac{1}{2}\pi$, and returned scattered radiation as that which is scattered backwards on the whole, *i.e.*, between the limits $\theta = \frac{1}{2}\pi$ and $\theta = \pi$. We shall denote the corresponding atomic and mass scattering coefficients by S_e, S_r, and σ_e, σ_r respectively.

Putting $\theta = \frac{1}{2}\pi$, $u = \dfrac{1}{\sqrt{2}}$ in (28), § 20, and using (30), § 21, we obtain for a single ring

$$\left.\begin{aligned} S_e &= \tfrac{1}{2} S_H n \left\{1 + \sum_1^{n-1} \left[f\left(\frac{r}{\sqrt{2}}\right) + g\left(\frac{r}{\sqrt{2}}\right)\right]\right\} \\ \text{where} \quad g(x) &= \tfrac{3}{8}\{S_1^2(x) - S_2^2(x)\}x^{-2} \\ \text{while} \quad r &= \frac{4\pi\sigma}{\lambda} \sin\frac{\pi i}{n} \end{aligned}\right\} \tag{36}$$

and $f(x)$ is given by (29), § 21, as before.

The value of S_r is clearly $S - S_e$, but for our purpose it is more convenient to use the difference $S_e - S_r$, which we shall denote by E. From (36) together with (29) and (30), § 21, we obtain

$$\left.\begin{aligned} E &= S_H n \sum_1^{n-1} h(r) \\ \text{where} \quad h(r) &= f\left(\frac{r}{\sqrt{2}}\right) + g\left(\frac{r}{\sqrt{2}}\right) - f(r), \quad r = \frac{4\pi\rho}{\lambda} \sin\frac{\pi i}{n} \end{aligned}\right\}. \tag{37}$$

Proceeding just as before in § 22 we obtain for the mass coefficients of an atom of atomic number N and atomic weight A

$$\eta = \frac{0{\cdot}40}{\mathrm{A}} \Sigma n \sum_{1}^{n-1} h(r), \qquad \sigma_e = \tfrac{1}{2}(\sigma+\eta), \qquad \sigma_r = \tfrac{1}{2}(\sigma-\eta), \qquad (38)$$

where σ is given by (34), § 22.

The coefficient η may be taken as an absolute measure of the fore and aft asymmetry of the atom on the same scale on which σ measures the total scattering; a relative measure would be afforded by the ratio $\eta : \sigma$, but the variation of this ratio with the wave-length differs so little from that of η itself that it is hardly worth while to introduce it in addition. The asymmetry is reckoned positive when η is positive, *i.e.*, when the emergent exceeds the returned scattered radiation, which is usually the case.

Numerical Values of the Coefficients.

24. For convenience sake, we collect the principal expressions which are of use for the purpose of comparison with experimental results, viz., (34), § 22, for the mass scattering coefficient σ, and (37) and (38), § 23, for the asymmetry coefficient η. They are

$$\left.\begin{aligned} \sigma &= \frac{0{\cdot}40}{\mathrm{A}}\left\{\mathrm{N} + \Sigma n \sum_{1}^{n-1} f(r)\right\}, & f(r) &= \{\mathrm{S}_0^2(r) + \tfrac{1}{2}\mathrm{S}_2^2(r)\}r^{-2} \\ \eta &= \frac{0{\cdot}40}{\mathrm{A}} \Sigma n \sum_{1}^{n-1} h(r), & h(r) &= f\left(\frac{r}{\sqrt{2}}\right) + g\left(\frac{r}{\sqrt{2}}\right) - fr, \\ & & g(r) &= \tfrac{3}{8}\{\mathrm{S}_1^2(r) - \mathrm{S}_2^2(r)\}r^{-2} \\ \text{where} & & r &= \frac{4\pi\rho}{\lambda}\sin\frac{\pi i}{n} \end{aligned}\right\}. \qquad (39)$$

The functions $f(r)$ and $h(r)$ are fundamental for the theory, whilst the function $g(r)$ is useful as an intermediary for purposes of calculation. Tables of these functions are given at the end of this paper for values of r from 0 to 10, and graphs of $f(r)$ and $h(r)$ are given in fig. 2 between the same limits.

We see that $f(r)$ has a sharp maximum equal to unity when $r = 0$, a low minimum equal to 0·046 when $r = 3{\cdot}1$, a comparatively low maximum equal to 0·076 nearly when $r = 4{\cdot}02$, a very low minimum equal to 0·002 when $r = 6{\cdot}1$, and so on, fluctuating between diminishing positive limits, and ultimately vanishing when r becomes infinitely great.

On the other hand, $h(r)$ vanishes when $r = 0$, has a fairly sharp maximum equal to 0·320 nearly when $r = 2{\cdot}1$, a small negative minimum equal to $-0{\cdot}022$ when $r = 4{\cdot}5$, a small maximum equal to 0·051 when $r = 6{\cdot}2$, and so on, fluctuating between diminishing alternately positive and negative limits and ultimately vanishing when r becomes infinitely great. We notice that negative asymmetry occurs for certain ranges of wave-length.

Comparison with Experiment.

25. In order to test the correctness of the theory by comparison with experimental results we must work out numerical values for particular arrangements of electrons, especially those that are likely to occur in nature. From the theoretical point of view the best examples to choose are the simplest, not only because they easily lead to definite numerical results, capable of experimental verification or otherwise, but also because the fundamental assumptions of the theory, given in §§ 2, 4, 19 and 22 are probably most nearly fulfilled in these cases. Of course, experimental difficulties may arise in realising the conditions needed for a test, such as the production of X- and γ-rays of sufficient intensity within the ranges of wave-length corresponding to the rapid changes of σ occurring near its principal maximum or to the maximum of η. These possible difficulties however do not concern us here, so that we shall choose the comparatively simple cases of helium and lithium as examples. We shall ignore the possibility that all their mobile electrons are arranged in separate rings, for this arrangement is extremely improbable from the theoretical point of view, and in any case would only give values for the scattering coefficients identical with those for hydrogen and therefore needing no further discussion.

26. *Helium.*—We make the usual assumptions $A = 4, N = n = 2$. In (39), § 24, there is but one sum, and this reduces to the single term $i = 1$; hence we have

$$\sigma = 0{\cdot}20\{1+f(r)\}, \qquad \eta = 0{\cdot}20\,a(r), \qquad r = 4\pi\rho/\lambda. \tag{40}$$

There is no need to give graphs, for obviously those given in fig. 2 will serve, if the vertical scale be reduced to one-fifth, and the constant amount 0·20 be added to give the graph of σ.

We see that for very short waves, for which $f(r)$ vanishes, σ approaches the value 0·20, one-half that of hydrogen, whilst for long waves, for which $f(r)$ is nearly equal to unity, σ approaches the value 0·40, that of hydrogen. For $r = 3{\cdot}1$ or $\lambda = 4{\cdot}05\rho$, σ is a minimum and equal to 0·209, for $r = 4{\cdot}2$ or $\lambda = 3\rho$, it is a maximum and equal to 0·215, so that for wave-lengths equal to 4ρ or less σ differs from the simple pulse theory's value 0·20 by $7\frac{1}{2}$ per cent. at most. On the other hand for $r = 1$, or $\lambda = 12{\cdot}6\rho$, $\sigma = 0{\cdot}34$ and differs from its theoretical maximum 0·40 by only 15 per cent.

In order to estimate what these wave-lengths correspond to in absolute measure we must form some idea of the value of ρ, the radius of the electron ring of helium. If we take Bohr's theory as a guide and assume that the angular momentum of each electron is $h/2\pi$, where h denotes Planck's constant, we find that $\rho = 0{\cdot}303$ Å.U., whilst β, the ratio of the velocity of the electrons to that of light, is $0{\cdot}0127$. Thus for the first minimum $\lambda = 1{\cdot}23$ Å.U., and for the 15 per cent. deficiency below the maximum $\lambda = 3{\cdot}82$ Å.U. Although our theory is quite independent of Bohr's, or any other particular theory of atomic structure, these numbers may serve to give some idea of suitable values of the wave-length for experimental tests. Moreover, since $r = 4\pi\rho/\lambda = 2\beta\nu/\omega$, the corresponding values of ν/ω are 122 and 39 respectively; these are so large that appreciable vitiation of the results owing to resonance is not to be feared.

Again, as regards the asymmetry, the maximum value of η is $0{\cdot}064$ and occurs for $r = 2{\cdot}1$, or $\lambda = 6\rho = 1{\cdot}82$ Å.U. This value of η corresponds to $\sigma = 0{\cdot}238$ and by (38), § 23, gives $\sigma_e = 0{\cdot}151$ and $\sigma_r = 0{\cdot}087$, so that the returned is about 58 per cent. of the emergent scattered radiation. Thus the greatest asymmetry is in the direction generally observed and is considerable even for so simple an atom as that of helium. Negative asymmetry is also possible, but it is very small, as we see from fig. 2. Its greatest value corresponds to $\eta = -0{\cdot}0044$, $\sigma = 0{\cdot}214$, $\sigma_e = 0{\cdot}100$ and $\sigma_r = 0{\cdot}104$, and occurs when $r = 4{\cdot}5$, or $\lambda = 2{\cdot}8\rho = 0{\cdot}84$ Å.U.; the corresponding returned is 104 per cent. of the emergent scattered radiation, but the difference is probably too small to be detected by experiment.

27. *Lithium.*—For this element two alternative arrangements of the three mobile electrons in the atom may be suggested: (1) all three in one ring, (2) two in one ring and one by itself in a co-axial ring. In each case we put $\mathrm{A} = 7{\cdot}06$, $\mathrm{N} = 3$; further we put $n = 3$ in (1), and $n = 2$ for the ring of radius ρ in (2), the radius of the other ring being immaterial, as its contribution to the scattering coefficients is independent of the wave-length and radius, as with the single electron ring of hydrogen. Thus we obtain from (39), § 24,

$$\left.\begin{array}{lll} \sigma_1 = 0{\cdot}170\{1+2f(r)\}, & \eta_1 = 0{\cdot}34h(r), & r = 2\sqrt{3}\pi\rho/\lambda \\ \sigma_2 = 0{\cdot}170\{1+\tfrac{2}{3}f(r)\}, & \eta_2 = 0{\cdot}113h(r), & r = 4\pi\rho/\lambda \end{array}\right\}. \qquad (41)$$

The graphs given in fig. 2 will again serve, provided that in case (1) we reduce the vertical scale to $0{\cdot}34$ of its value and add the constant amount $0{\cdot}17$ to give σ, whilst in case (2) we reduce the vertical scale to $0{\cdot}113$ of its value and add the same constant amount $0{\cdot}17$ to give σ.

We see that for very short waves $\sigma = 0{\cdot}17$, and $\eta = 0$ in both cases, in agreement with the simple pulse theory. But for long waves σ_1 approaches the maximum value $0{\cdot}51$, considerably greater than the value of σ for hydrogen, whilst σ_2 only approaches the maximum value 0·283, much less than that for hydrogen.

Again, the maximum value of η_1 is 0·109, gives $\sigma = 0{\cdot}205$, $\sigma_r/\sigma_e = 31$ per cent., and occurs when $r = 2{\cdot}1$ or $\lambda = 6\rho$, whilst the maximum value of η_2 is only 0·036, $\sigma = 0{\cdot}192$, $\sigma_r/\sigma_e = 68$ per cent., and occurs for $r = 2{\cdot}1$ or $\lambda = 6\rho$. Thus we see that both the scattering coefficient and maximum asymmetry are so much greater for the first arrangement than for the second that experiment should enable us to decide between them.

In the case of lithium, estimates of wave-lengths are rendered uncertain by the failure of Bohr's theory to provide a satisfactory explanation of the spectrum: nevertheless they are not likely to be so far wrong as to be without value. For the first alternative Bohr's theory gives $\rho = 0{\cdot}22$ Å.U., $\beta = 0{\cdot}0176$, so that the wave-length for maximum asymmetry is 1·14 Å.U., The first minimum of σ is given by $r = 3{\cdot}1$, or $\lambda = 3{\cdot}51\,\rho = 0{\cdot}77$ Å.U., whilst 15 per cent. deficiency from the maximum corresponds to $f(r) = 0{\cdot}775$, $r = 0{\cdot}866$, or $\lambda = 12{\cdot}6\,\rho = 2{\cdot}77$ Å.U. The corresponding values of ν/ω are 140 and 39, so that resonance can hardly be appreciable.

The second alternative is much more uncertain, for we do not know whether the two rings are in the same plane or not, nor do we know which has the greater radius. Consequently we may content ourselves with the rough approximation obtained by neglecting the action of the outer ring on the inner one, and treating the latter as part of the nucleus in estimating its action on the former. Then the radius of the outer ring will be underestimated, that of the inner overestimated, because the repulsion between them is underestimated.

In subcase (*a*), with the ring of two electrons outside, this ring is practically the helium ring, so that we have $\rho = 0{\cdot}303$ Å.U., $\beta = 0{\cdot}0127$, as in § 26. The maximum asymmetry is given by $\lambda = 6\rho = 1{\cdot}82$ Å.U., the first minimum of scattering by $r = 3{\cdot}1$, $\lambda = 4{\cdot}05\,\rho = 1{\cdot}23$ Å.U., $\nu = 122\,\omega$, and the 15 per cent. deficiency below maximum scattering by $f(r) = 0{\cdot}625$, $r = 1{\cdot}165$, $\lambda = 10{\cdot}7\,\rho = 3{\cdot}24$ Å.U., $\nu = 46\,\omega$.

In subcase (*b*), with the two-electron ring inside and practically subject to the attraction of three positive charges, we find that $\rho = 0{\cdot}19$ Å.U., $\beta = 0{\cdot}0200$. The maximum asymmetry is given by $\lambda = 6\rho = 1{\cdot}14$ Å.U., the first minimum of scattering by $\lambda = 4{\cdot}05\,\rho = 0{\cdot}77$ Å.U., $\nu = 77\,\omega$, and the 15 per cent. deficiency below maximum scattering by $\lambda = 10{\cdot}7\,\rho = 2{\cdot}03$ Å.U., $\nu = 29\,\omega$.

Thus for lithium the important range of wave-lengths may be regarded as extending from 0·8 Å.U. to 3·2 Å.U., and the values of ν/ω are so great that appreciable vitiation by resonance is hardly likely.

28. The examples of helium and lithium, discussed in §§ 26 and 27, are sufficient to indicate how experiments on the scattering of X- and γ-rays may be used with the twofold object, first, of testing the correctness of the theory of scattering here developed, and with it the truth of the fundamental principles of the electron ring theory on which it is based, and, secondly, of determining the arrangement of the electrons in the atom, if and when the first test has proved satisfactory. It is important to bear in mind that the estimates of wave-lengths given above are only intended to serve as rough guides to the experimenter; their correctness, or otherwise, has no importance for the theory of scattering as such, but can only serve as a test of Bohr's theory of atomic structure. If the theory of scattering survives the test of experiments on the simplest atoms, such as those of hydrogen and helium, further measurements of the scattering by more complex atoms, such as those of lithim, carbon, and so on, will enable us to obtain information respecting the arrangement of the electrons in these atoms. The character of the variation of scattering and asymmetry with the wave-length depends upon the number of rings and the number of electrons in each ring, whilst the position of the maxima and minima depends upon the radii of the rings.

Very few experimental results are available for comparison as yet, and no absolute measurements at all on hydrogen and helium, which are so necessary for a test of the theory. Aurén* has given a very valuable series of measurements relative to water, from which it appears that the *atomic* scattering coefficient of lithium is about four times that of hydrogen for $\lambda = 0{\cdot}36$ Å.U. This gives $\sigma = 0{\cdot}227$, with an error which may be as much as 10 or 20 per cent., because of the uncertainty in Aurén's value for the coefficient of hydrogen; for this standard number has to be calculated from water and carbon compounds and necessarily appears as the difference between nearly equal numbers. The value of σ is much greater than the simple pulse theory's value, 0·17, and is well within the limits given in § 27, but it is too unreliable to furnish a reliable estimate for ρ or to decide between the two arrangements discussed there.

29. We must now consider certain experimental results obtained by Ishino† for very hard γ-rays scattered by aluminium, iron, and lead, bearing in mind that our theory can only be applied to atoms as complex as these with a very rough approximation owing to the comparatively large values of β that are to be expected for some of their rings.

In the first place, Ishino and Madsen before him, found very great asymmetry coupled with a very small scattering coefficient, viz., a ratio

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* 'Phil. Mag.,' Ser. 6, vol. 37, p. 165 (1919).

† 'Phil. Mag.,' Ser. 6, vol. 33, p. 129 (1917).

$\sigma_r : \sigma_e$ equal to 0·15 and a mass scattering coefficient of the order 0·04 for aluminium and 0·03 for lead. Thus we obtain for aluminium $\sigma = 0{\cdot}04$, $\eta = 0{\cdot}03$, and for lead $\sigma = 0{\cdot}03$, $\eta = 0{\cdot}022$, so that η is about three-quarters of σ for each element. The asymmetry is greater than any we have discussed, but is not so great that it cannot be accounted for by the presence of rings of many electrons. In § 27 we found that a ring of three electrons gave a value of η one-half that of σ, so that a ring of seven electrons, such as Vegard assumes for aluminium, may easily give the requisite asymmetry.

But the smallness of the mass scattering coefficient remains unexplained by the present theory, which gives the simple pulse theory's value as a lower limit. The exceptionally small scattering coefficients observed by Ishino are, however, accompanied by a change of type, a softening of the rays, pointing to an emission of rays accompanying the scattered rays, but of greater wave-length, a fluorescence in other words. May not these be due to *free* vibrations of the electrons, generated by the incident radiation and accompanying the *forced* vibrations to which scattering is due? There is *reason for supposing that some of the free vibrations of the electron rings* are but slightly damped and therefore comparatively persistent, otherwise interference with long path differences would be impossible. Moreover, we know from ordinary dynamics that the initiation of a forced vibration is on the average accompanied by the production of a free vibration, but when the incident vibration is undamped, only a small fraction of its energy is used in generating free vibrations on the average. When, however, the incident vibration is strongly damped, a large fraction of its energy may be used in this way, and a correspondingly smaller fraction scattered. This problem is too long to be discussed here, but is under investigation at present.

The Ring Electron.

30. Just after the completion of the preceding part of this investigation I became acquainted with the full text of the paper on "The Size and Shape of the Electron," by A. H. Compton,* in which he considers the scattering of the X- and γ-rays by the flexible ring electron amongst other topics. Evidently this electron may be regarded as the limit of an electron ring when the number of the electrons becomes infinitely great, whilst the total charge and mass remain finite. Hence we should obtain expressions for the scattering coefficients of the ring electron as limiting values of our expressions for the electron ring, especially (25), § 19, and (29), § 21.

For this purpose we suppose n to become very large and replace e by e/n, m by m/n, $\pi i/n$ by ϕ, and $1/n$ by $d\phi/\pi$. The summations with respect to

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* 'Phys. Rev.,' vol. 14, p. 20 (1919).

i are taken most conveniently from 0 to $n-1$, *i.e.*, including the first term, previously separated from the rest, and become integrations with respect to ϕ from 0 to π.

Thus we obtain from (25), § 19,

$$\left.\begin{aligned} \bar{s} &= \frac{e^4}{2\pi c^4 m^2}(1+\cos^2\theta)\int_0^\pi \frac{\sin(4\tau\sin\frac{1}{2}\theta\sin\phi)}{4\tau\sin\frac{1}{2}\theta\sin\phi}\,d\phi \\ &= \frac{e^4}{2c^4m^2}(1+\cos^2\theta)\int_0^1 \mathrm{J}_0(4v\tau\sin\tfrac{1}{2}\theta)\,dv \\ &= \frac{e^4}{2c^4m^2}(1+\cos^2\theta)\sum_0^\infty(-1)^s\frac{(2\tau\sin\frac{1}{2}\theta)^{2s}}{(s\,!)^2(2s+1)} \end{aligned}\right\} \tag{42}$$

where $\tau = 2\pi\rho/\lambda$ as before.

31. The series represents Compton's ratio $\mathrm{L}^2\mathrm{I}_{\mathrm{L},\,\theta}:\mathrm{I}$, and can be compared with his series (20),* where a is used instead of our ρ for the radius of the ring. The form of the series is the same, but the coefficients are different, on account of a difference in the mode of averaging for the varying orientations of the rings in an assemblage. In our fig. 1, § 7, the pole of the ring, z, is fixed by its angular distance zV, or γ, from the fixed point V and by the angle zVU, or Ω, between the great circle zV and the fixed great circle UV; and we have taken the probability of the orientation (γ, Ω) to be $\sin\gamma d\gamma d\Omega/4\pi$. On the other hand, Compton determines the position of the ring by the angular distance TV, or α, from the fixed point V of the intersection T of the great circle xyT (the plane of the ring) with the fixed great circle UV and by the angle yTV, or β, between the two great circles; and he takes the probability of the orientation (α, β) to be $d\alpha d\beta/\pi^2$. It is easy to prove from the spherical triangle zVT that $\sin\gamma d\gamma d\Omega = \sin\beta d\alpha d\beta$. The difference amounts to this: with us the poles z of the rings are distributed uniformly over the unit sphere, whilst with Compton the point T is distributed uniformly over the great circle UV, and at the same time the plane of the ring xOy is distributed uniformly about the nodal line OT. The difference is not, however, of fundamental importance.

32. To obtain $\mathrm{S}(\theta)$, the whole scattering forwards through a cone of semivertical angle θ, we multiply (42), § 30, by $2\pi\sin\theta d\theta$, or $8\pi u du$, where $u = \sin\frac{1}{2}\theta$, and integrate as before. Thus we find

$$\mathrm{S}(\theta) = \frac{4\pi e^4}{c^4m^2}\sum_0^\infty(-1)^s\frac{(2\tau u)^{2s}}{(s\,!)^2(2s+1)}\left(\frac{u^2}{s+1}-\frac{2u^4}{s+2}+\frac{2u^6}{s+3}\right). \tag{43}$$

The total scattering is given by $\theta = \pi$, or $u = 1$; we obtain

$$\mathrm{S} = \frac{8\pi e^4}{3c^4m^2}\sum_0^\infty(-1)^s\frac{3(s^2+3s+4)(2\tau)^{2s}}{2(s\,!)\,(s+3\,!)\,(2s+1)}. \tag{44}$$

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* *Loc. cit.*, p. 42.

This can be compared, with Compton's series (21),* which has the same form as (44), but different coefficients for the reason above assigned.

As regards asymmetry, we obtain Florance's emergent scattered radiation scattered in the forward direction by putting $\theta = \frac{1}{2}\pi$, or $u = \frac{1}{\sqrt{2}}$ in (43); thus we have

$$S\left(\tfrac{1}{2}\pi\right) = \frac{4\pi e^4}{3c^4m^2}\sum_0^\infty (-1)^s \frac{3(s^2+5s+8)(\sqrt{2}\tau)^{2s}}{4(s\,!)(s+3\,!)(2s+1)}. \tag{45}$$

The coefficients in the series (44) and (45) are arranged so that the first term is unity in each case; then the coefficient of τ^{2s} in (45) is

$$(s^2+5s+8)/(s^2+3s+4)\,2^{s+1}$$

times that in (44). It follows that the ratio $S(\frac{1}{2}\pi) : S$, which is one-half when the frequency vanishes, or $\tau = 0$, increases with τ to a maximum, and then probably alternates between ever-narrowing limits as the successive terms of the series predominate in turn, in somewhat the same manner as the mass asymmetry coefficients in fig. 2.

33. The series (44) and (45) are convenient when τ is small, or λ/ρ large, but for large values of τ, or small values of λ/ρ, the computation becomes very laborious owing to the great number of terms required. In this case it is better to use (29), § 21, and (36) § 23, or their equivalents (32), § 22, and (37), § 23. We obtain, by proceeding to the limit for $n \rightarrow \infty$, as in § 30,

$$\left.\begin{aligned} S &= \frac{8\pi e^4}{3c^4m^2}\frac{1}{\pi}\int_0^\pi f\left(\frac{4\pi\rho}{\lambda}\sin\phi\right)d\phi = \frac{8\pi e^4}{3c^4m^2}\frac{2}{\pi}\int_0^{2\tau}\frac{f(r)\,dr}{\sqrt{(4\tau^2-r^2)}} \\ E &= \frac{8\pi e^4}{3c^4m^2}\frac{1}{\pi}\int_0^\pi h\left(\frac{4\pi\rho}{\lambda}\sin\phi\right)d\phi = \frac{8\pi e^4}{3c^4m^2}\frac{2}{\pi}\int_0^{2\tau}\frac{h(r)\,dr}{\sqrt{(4\tau^2-r^2)}} \end{aligned}\right\}, \tag{46}$$

where $\tau = 2\pi\rho/\lambda$ as before. S is, of course, the total scattering coefficient of the ring electron as before, whilst E is its asymmetry coefficient, defined as in § 23, and equal to the excess of the "emergent scattered radiation," $S(\frac{1}{2}\pi)$, over the "returned scattered radiation," $S - S(\frac{1}{2}\pi)$, each being reckoned per unit intensity of the incident radiation.

A graphic method of computation could be based on the second type of integral by making use of the graphs of the functions $f(r)$ and $h(r)$ given in fig. 2, but the vanishing factor $\sqrt{(4\tau^2-r^2)}$ would cause difficulties near the upper limit. On the whole, the best method is based on the first type of integral: for a given value of τ, or $2\pi\rho/\lambda$, the argument $2\tau\sin\phi$ is calculated for a number of equidistant values of ϕ from 0 to $\frac{1}{2}\pi$; corresponding values of f and h are found from the Tables at the end of this paper, and then one of the usual summation formulæ is applied. When the interval of ϕ is π/n, where n is an even integer, this amounts to approximately the same thing as calculating S, or E, as the case may be, for a ring of n electrons and dividing the result by n^2. The greater n is, the better the approximation will be. Put briefly we may say:

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* *Loc. cit.*, p. 42.

The scattering, or asymmetry, of the ring electron of radius ρ, charge e and mass m, is the limit, when n is infinitely great, of n^{-2} times the scattering, or asymmetry, of a ring, of radius ρ, of n equidistant electrons of the usual type, each of charge e and mass m.

34. The results of most importance for a comparison of the three theories discussed in the preceding pages may be summarised in the following Table, where the total scattering is for convenience sake expressed in terms of S_H, or $8\pi e^4/3c^4m^2$, the total scattering of the hydrogen atom on the simple pulse theory, as a unit. N denotes the atomic number, taken to be the total number of mobile electrons in the atom, whether ring electrons or electrons of the usual type. For the electron ring theory n, as before, denotes the number of electrons of the usual type in any one ring, and the sign of summation refers to the several rings, so that $N = \Sigma n$.

Theory.	Maximum total scattering. (Long waves.)	Minimum total scattering. (Short waves.)	Maximum asymmetry.
Simple Pulse	N for all wave lengths.		0
Electron Ring	Σn^2	N	Increasing as n increases.
Ring electron............	N	0	Always large.

The corresponding values of the mass scattering coefficients relative to hydrogen are obtained from those of the relative atomic scattering coefficients given in the second and third columns by dividing by A, the atomic weight of the element considered.

The Table enables us to indicate experiments which might be crucial in deciding between the several theories. For example, for hydrogen we take $N = n = 1$; according to the simple pulse and electron ring theories the total scattering has the same value unity for all wave-lengths, and there is no asymmetry, but according to the ring electron theory the total scattering only has the value unity for long waves and diminishes to zero for short ones, whilst the maximum asymmetry is as large as that found by Ishino for aluminium, iron, and lead, but probably occurs at a longer wave-length. If the scattering from hydrogen, whether gas under pressure or liquid, can be observed at all, even quite rough measurements should be decisive for or against the ring electron theory, but could not decide between the other two theories.

Again, for helium we may put N = n = 2; according to the simple pulse theory the relative total scattering is 2 for all wave-lengths, and there is no asymmetry; according to the electron ring theory the relative total scattering is 4 for very long waves and diminishes to 2 for very short ones, and there is appreciable asymmetry, but much less than that found by Ishino for aluminium and the other metals; lastly, according to the ring electron theory, the relative total scattering is 2 for very long waves and diminishes to zero for very short ones, and there is large asymmetry, comparable with that found by Ishino for aluminium.

Similar results hold for other elements: speaking generally, the simple pulse theory gives relative total scattering equal to the atomic number independently of the wave-length, and no asymmetry; the electron ring theory gives exceptionally large relative scattering for very long waves diminishing to the simple pulse theory's value as a *lower* limit for very short waves, together with appreciable, but relatively small, maximum asymmetry for light atoms and large maximum asymmetry for heavy ones; and lastly, the ring electron theory gives the simple pulse theory's value as an *upper* limit of the relative total scattering for very long waves and values diminishing to zero for very short waves, together with large maximum asymmetry of the same order for all elements, whether light or heavy.

35. The experimental data available for a comparison are few and contradictory.

The relative measurements of Aurén on the total scattering for wave-lengths between 0·34 and 0·38 Å.U. give values much greater than the simple pulse theory's value even for *light* elements, *e.g.*, 0·227 instead of 0·17 for lithium, already mentioned in § 28. Aurén's values include absorption and therefore are subject to vitiation by resonance; but no characteristic radiation of lithium has been observed near his range of wave-lengths, although the J radiation of carbon, originally discovered by Barkla, is not far away. Since 0·17 is the *upper* limit of the total scattering of lithium according to the ring electron theory, this result, as well as others of Aurén, is against that theory, but it is not decisive because the possibility of vitiation by resonance is not excluded.

On the other hand, as we saw in §§ 29 and 35, Ishino found values of the total scattering of aluminium, iron and lead for hard γ-rays of a wave-length about 0·07 Å.U., which were only one-quarter of those to be expected according to the simple pulse theory and also the electron ring theory for very short waves. Compton shows that values of this order are to be expected for a ring electron of radius 0·02 Å.U.; they cannot be explained at all by the simple pulse theory, nor by the electron ring theory developed above for *light* elements, so long as the γ-rays are regarded as undamped. But two possible explanations may be suggested. In the first place aluminium, iron, and lead are not *light* elements in the sense of the present investigation, in which the effect of the speed of the electrons has been neglected. We have some reason for supposing that some of their mobile electrons move with speeds quite appreciable compared with that of light, so that their mass is greater than m, that of a slowly moving electron, and the effect of the magnetic field of the incident radiation also must be taken into account. The effect of increase of mass is to diminish the total scattering coefficient; that of the magnetic field requires investigation. Secondly, damping of the incident radiation may be expected to diminish the scattering for the reasons already given in § 29, but the investigation alluded to there, so far as it has gone, shows that it can have no effect in the case of a ring of one electron (hydrogen). It may be urged that damping is irreconcilable with the production of fine spectrum lines, but it should be borne in mind that the resolving power of X-ray spectrographs is far below that of optical instruments, and the phase differences reached are correspondingly smaller, and this seems to be still more true for γ-rays, such as those used by Ishino. Thus an amount of damping would be possible for hard γ-rays, which would be quite out of the question for light. Moreover, it is suggestive that the deficiencies of the total scattering below the simple pulse theory's value observed for carbon and aluminium by Barkla and White* are very much smaller than those observed by Ishino, commence at wave-lengths from 0·3 to 0·2 Å.U., and increase as the wave-length diminishes. In any case until the investigations on the effects of damping of the incident radiation and of the speed of the electrons have been completed, Ishino's experiments cannot be regarded as decisive against the electron ring theory, although they are strongly in favour of the ring electron theory.

[*Note added December* 4.—The damping investigation has been completed and gives the following results:—For values of r less than 1·5 slight damping *diminishes* the scattering function $f(r)$, but for greater values it *increases* it, but the changes are very small for any admissible amount of damping. Large damping almost certainly diminishes the scattering to the simple pulse theory's value, but not below. Hence damping cannot explain Ishino's low values. The effect of electron speeds comparable with that of light is under consideration.]

36. In order that experiments on scattering should lead to definite theoretical results, the following conditions ought to be approximated to as

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* 'Phil. Mag.,' Ser. 6, vol. 34, p. 270 (1917).

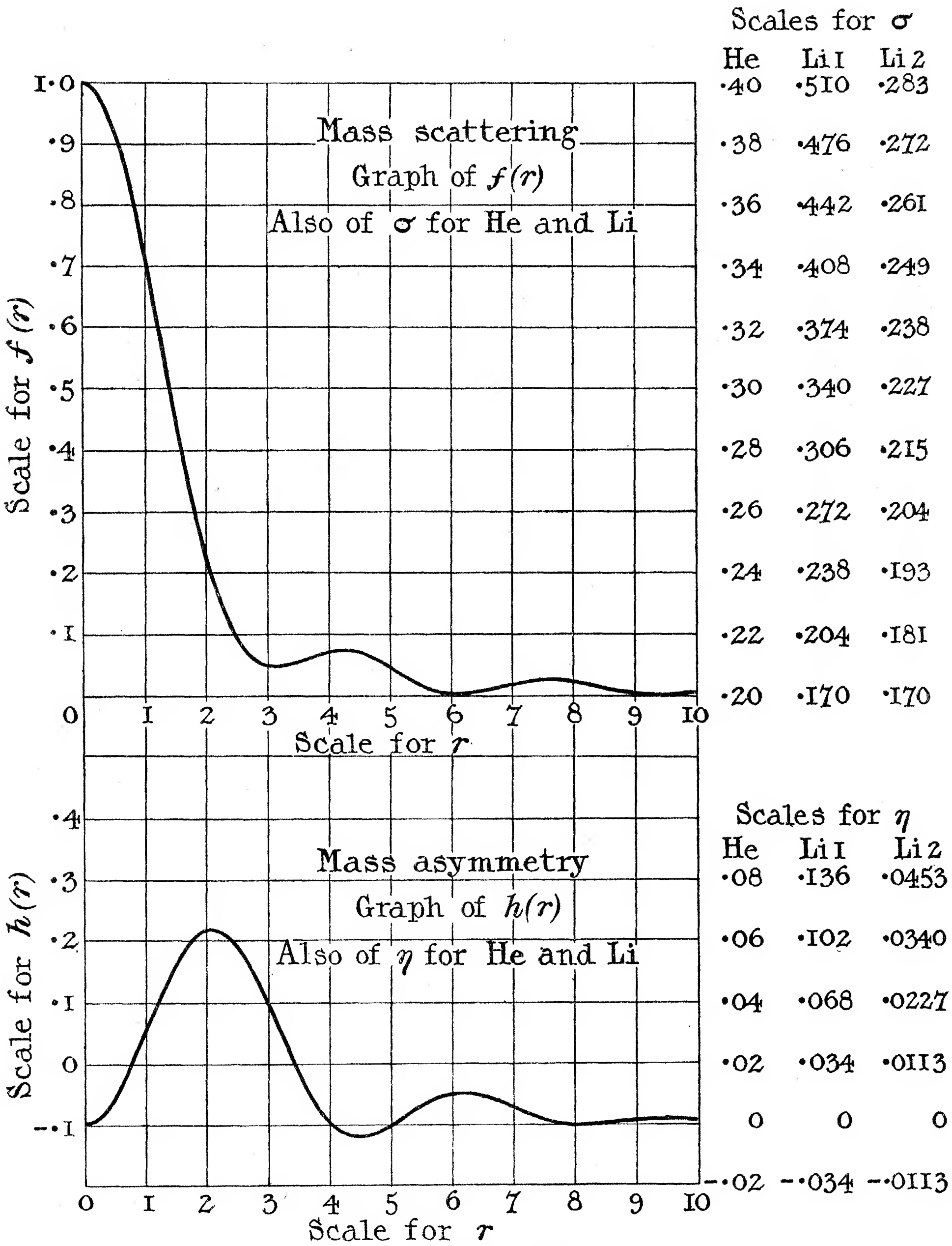

Fig. 2.—Coefficients of He and Li.

nearly as may be consistent with the use of an incident radiation of sufficient intensity to enable measurements of reasonable accuracy to be made: (1) the incident radiation should be monochromatic; (2) the range of wave-lengths should be wide ; (3) the scattered radiation should be sifted from any accompanying resonance or fluorescence radiation of different wave-lengths ; (4) contamination of light elements with heavier impurities should be avoided on account of the disproportionate effect of the heavier elements on the scattering for the longer waves.

Experiments on these lines on the absolute scattering in various directions of X- and γ-rays by hydrogen, as well as other light elements, either absolute, or relative to hydrogen, for suitable ranges of wave-lengths, may be expected to give valuable information respecting the nature of the electron, the arrangement of the electrons in light atoms, and the nature of X- and γ-rays,

Tables of the Functions $f(r)$, $g(r)$, $h(r)$.

r.	$f(r)$.	$g(r)$.	$h(r)$.	r.	$f(r)$.	$g(r)$.	$h(r)$.
0·2	0·9867	0·0017	0·0068	5·2	0·0338	+0·0020	+0·0111
0·4	0·9479	0·0064	0·0286	5·4	0·0225	0·0063	0·0246
0·6	0·8859	0·0137	0·0624	5·6	0·0131	0·0091	0·0367
0·8	0·8049	0·0228	0·1049	5·8	0·0064	0·0102	0·0454
1·0	0·7100	0·0326	0·1529	6·0	0·0029	0·0100	0·0503
1·2	0·6070	0·0419	0·2008	6·2	0·0023	0·0082	0·0511
1·4	0·5019	0·0497	0·2452	6·4	0·0043	0·0058	0·0484
1·6	0·4003	0·0551	0·2815	6·6	0·0082	0·0024	0·0426
1·8	0·3072	0·0575	0·3063	6·8	0·0130	−0·0006	0·0344
2·0	0·2264	0·0564	0·3183	7·0	0·0178	−0·0033	0·0262
2·2	0·1604	0·0518	0·3163	7·2	0·0218	−0·0055	0·0178
2·4	0·1103	0·0443	0·2990	7·4	0·0244	−0·0065	0·0102
2·6	0·0758	0·0344	0·2728	7·6	0·0252	−0·0066	0·0045
2·8	0·0554	0·0231	0·2353	7·8	0·0241	−0·0058	0·0010
3·0	0·0468	0·0114	0·1921	8·0	0·0215	−0·0042	−0·0008
3·2	0·0470	0·0022	0·1462	8·2	0·0176	−0·0022	−0·0010
3·4	0·0526	−0·0095	0·1011	8·4	0·0132	−0·0001	+0·0007
3·6	0·0607	−0·0169	0·0598	8·6	0·0088	+0·0019	0·0028
3·8	0·0685	−0·0218	0·0252	8·8	0·0049	0·0033	0·0055
4·0	0·0740	−0·0235	0·0016	9·0	0·0022	0·0042	0·0080
4·2	0·0757	−0·0228	−0·0149	9·2	0·0007	0·0044	0·0095
4·4	0·0733	−0·0198	−0·0219	9·4	0·0005	0·0042	0·0104
4·6	0·0669	−0·0150	−0·0201	9·6	0·0016	0·0029	0·0108
4·8	0·0574	−0·0093	−0·0141	9·8	0·0035	0·0015	0·0105
5·0	0·0459	−0·0034	−0·0025	10·0	0·0060	0·0000	0·0093

In conclusion, I wish to thank Prof. Sir J. Larmor for his interest in the work, and his valuable criticisms and suggestions.